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Optimal Control of Linear Dynamical System with Intermediate Phase Constraints

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Table of contents

Table of contents

Table of contents

Table of contents

Introduction

Optimal control theory is an important area of applied mathematics, developed to find optimal way to control management systems, overcome the arduous tasks, predict and control future events and finally to optimize a certain criteria.

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In the class of piecewise constant controls, consider the following optimal control problem:

$$J(u) = c_1' x(t^*) + \int_0^{t^*} c_2'(t) u(t) dt \longrightarrow \max, \quad (1a)$$

$$\dot{x} = Ax + Bu + r, \quad x(0) = x_0, \quad (1b)$$

$$g_*(s) \leq H(s) x(t_s) \leq g^*(s), \quad (1c)$$

$$d^- \leq u(t) \leq d^+, \quad t \in T = [0, t^*], \quad (1d)$$

where:
$A = A(K,K)$, $B = B(K,J)$, $H(s)H(I(s),K)$; $g_*(s) = g_*(I(s))$, $g^* = g^*(I(s))$, $d^- = d^-(J)$, $d^+ = d^+(J)$;
$x(t), c_1 \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$. $K = \{1 \ldots n\}$, $J = \{1 \ldots r\}$, $I(s) = \{1, \ldots, m_s\}$, $s \in S = \{1, \ldots, m\}$.

Using the Cauchy formula

$$x(t) = F(t)\left[x_0 + \int_0^t F^{-1}(\tau)(Bu(\tau) + r(\tau))d\tau\right], \quad t \in T, F(t) = \exp(At),$$

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we can write the dynamic optimization problem (1) in the following equivalent form:

$$\begin{cases} J(u) = c_1' F(t^*) x_0 + \int_0^{t^*} c'(t) u(t) dt + \int_0^{t^*} c_3'(t) r(t) dt \longrightarrow \max, \\ \overline{g}_*(s) \le \int_0^{t_s} \varphi(s,t) u(t) dt \le \overline{g}^*(s), s \in S, \\ d^- \le u(t) \le d^+, \quad t \in T = [0, t^*], \end{cases} \tag{2}$$

where:

$c'(t) = c_1' F(t^*) F^{-1}(t) B + c_2'(t)$, $c_3'(t) = c_1' F(t^*) F^{-1}(t)$, $\overline{g}_*(s) = g_*(s) - H(s) F(t_s)[x_0 + \int_0^{t_s} F^{-1}(t) r(t) dt]$, $\overline{g}^*(s) = g^*(s) - H(s) F(t_s)[x_0 + \int_0^{t_s} F^{-1}(t) r(t) dt]$;

$$\varphi(s,t) = \begin{cases} H(s) F(t_s) F^{-1}(t) B, & \text{if } 0 \le t \le t_s, \\ 0, & \text{if } t > t_s, \end{cases} \tag{3}$$

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Construction of support

In the adaptive method, the principal tool is the support. In order to define it, let us choose a subset $S_{sup} \in S$, and for each $s \in S_{sup}$ we choose an arbitrary subset $I_{sup}(s) \in I(s)$, and we form $I_{sup} = \{I_{sup}(s), \ \ s \in S_{sup}\}$, with $|I_{sup}| = p \leq \sum_{s \in S} m_s$.

On the interval T, we choose also a subset of isolated moments:

$$T_{sup} = \{t_k, k \in K_{sup}\}, \ \ K_{sup} = \{1, \dots, k^*\}, \ |K_{sup}| \leq p.$$

For each moment $t_k \in T_{sup}$, we associate a set of indices $J_k \subset J$, such that $\sum_{k \in K_{sup}} |J_k| = p$.

We assume $J_{sup} = \{J_k, k \in K_{sup}\}$ and $Q_{sup} = \{I_{sup}, J_{sup}, T_{sup}\}$, and form the $p \times p$ matrix:

$$\varphi_{sup} = \varphi(Q_{sup}) = (\varphi_{ij}(s, t_k), \ i \in I_{sup}(s), \ \ s \in S_{sup}, \ j \in J_k, \ k \in K_{sup}). \tag{4}$$

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Definition 1

A piecewise constant control $u(t), t \in T$, is said to be an admissible control if it satisfies the constraints (1b) and (1c).

Definition 2

An admissible control $u^0(t), t \in T$, is called an optimal control if

$$J(u^0) = \max J(u). \tag{5}$$

Definition 3

Moreover, we call ϵ-optimal (or suboptimal) control any admissible control $u^\varepsilon(t), t \in T$, satisfying the inequality:

$$J(u^0) - J(u^\varepsilon) \leq \varepsilon, \tag{6}$$

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The set $Q_{sup} = \{I_{sup}, J_{sup}, T_{sup}\}$ is called a support of the problem (1) if $det\varphi_{sup} \neq 0$.

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The pair $\{u, Q_{sup}\}$ formed by an admissible control u and a support Q_{sup} is called an admissible support control.

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Optimality criteria

Table of contents

Increment formula of the functional

Let $\{u, Q_{sup}\}$ be a support control of problem (1) and consider another admissible control $\overline{u}(t) = u(t) + \Delta u(t)$ and its corresponding trajectory $\overline{x}(t) = x(t) + \Delta x(t)$, $t \in T$.

The increment formula of the functional is expressed as follows:

$$\begin{aligned}
\Delta J(u) &= J(\overline{u}) - J(u) \\
&= c_1' F(t^*)x_0 + \int_0^{t^*} (c'(t)\overline{u}(t) + c_3'(t)r(t))dt \\
&- c_1' F(t^*)x_0 - \int_0^{t^*} (c'(t)u(t) + c_3'(t)r(t))dt \\
&= \int_0^{t^*} c'(t)(\overline{u}(t) - u(t))dt = \int_0^{t^*} c'(t)\Delta u(t)dt. \qquad (7)
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Increment formula of the functional

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Increment formula of the quality criterion

- Construct the potential discret function:

$$y(s) = (y_i(s), i \in I(s)) = (y(I_{sup}(s)), y(I_c(s)) = 0), s \in S, \quad (8)$$

with $y'(I_{sup}) = (y_i(s), i \in I_{sup}(s), \quad s \in S_{sup}) = c'_{sup}\varphi^{-1}_{sup}$.

where: $c_{sup} = (c_j(t_k), j \in J_k, k \in K_{sup})$, $I_c(s) = I(s) \backslash I_{sup}(s), \quad s \in S$.

- Define the cocontrol E:

$$E'(t) = -\psi'(t)B - c'_2(t), \; t \in T. \quad (9)$$

where: $\psi(t)$ is the cotrajectory of the adjoint system:

$$\dot{\psi} = -A'\psi, \quad \psi(t^*) = c_1, \quad (10)$$

and it jumps at the intermediate moments as follows:

$$\psi(t_s - 0) = \psi(t_s + 0) - H'(s)y(s).$$

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Increment formula of the quality criterion

Using these relations, the increment of the functional becomes:

$$\Delta J(u) = \sum_{s \in S_{sup}} \sum_{i \in I_{sup}(s)} y_i(s) \upsilon_i(s) - \int_0^{t^*} E'(t) \Delta u(t) dt. \quad (11)$$

where: $H(s)\Delta x(t_s) = \upsilon(s)$.

Therefore, it is clear that the maximum of this increment of the functional under the constraints :

$$\begin{cases} g_{*i}(s) - H_s(i,K)x(t_s) \leq \upsilon_i(s) \leq g_i^*(s) - H_s(i,K)x(t_s), & i \in I_{sup}, \\ d^- - u(t) \leq \Delta u(t) \leq d^+ - u(t), & t \in T, \end{cases} \quad (12)$$

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Increment formula of the quality criterion

is equal to:

$$\beta(u, Q_{sup}) = \sum_{j=1}^{r} \left[\int_{T_j^+} E_j(t)(u_j(t) - d_j^-)dt + \int_{T_j^-} E_j(t)(u_j(t) - d_j^+)dt \right]$$
$$+ \sum_{s \in S_{sup}} \sum_{y_i(s)<0, i \in I_{sup}(s)} y_i(s)\upsilon_i^-(s) + \sum_{s \in S_{sup}} \sum_{y_i(s)>0, i \in I_{sup}(s)} y_i(s)\upsilon_i^+(s$$

where

$$H_s = H(s), \quad T_j^+ = \{t \in T : E_j(t) > 0\}, \quad T_j^- = \{t \in T : E_j(t) < 0\}, \quad j \in J,$$

and

$$\upsilon^-(I(s)) = (\upsilon_i^-(s), i \in I(s)) = g_*(s) - H(s)x(t_s), s \in S,$$

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The number $\beta(u, Q_{sup})$ is called the suboptimality estimate of the support control $\{u, Q_{sup}\}$

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$$H_s = H(s),\ \ T_j^+ = \{t \in T : E_j(t) > 0\},\ \ T_j^- = \{t \in T : E_j(t) < 0\},\ \ j \in J,$$

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Table of contents

Optimality criteria

Theorem 1 "Optimality criteria"

Let (u, Q_{sup}) be a support control of the problem (1). The following relations

$$\begin{cases} E_j(t) \geq 0, & \text{if } u_j(t) = d_j^-, \\ E_j(t) \leq 0, & \text{if } u_j(t) = d_j^+, \\ E_j(t) = 0, & \text{if } d_j^- < u_j(t) < d_j^+, \quad t \in T, \quad j \in J; \\ & \\ y_i(s) \geq 0, & \text{if } H_s(i, K)x(t_s) = g_i^*(s), \\ y_i(s) \leq 0, & \text{if } H_s(i, K)x(t_s) = g_{*i}(s), \\ y_i(s) = 0, & \text{if } g_{*i}(s) < H_s(i, K)x(t_s) < g_i^*(s), \quad i \in I_{sup}(s), \quad s \in S_{sup}, \end{cases} \tag{13}$$

are sufficient, and in the case of nondegeneracy also necessary, for the optimality of the support control (u, Q_{sup}).

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are sufficient, and in the case of nondegeneracy also necessary, for the optimality of the support control (u, Q_{sup}).

Construction of the algorithm

Construction of the Algorithm

Then let $\varepsilon \geq 0$ and $\{u, Q_{sup}\}$ be an initial support control. The aim of the algorithm is to construct a suboptimal control u^{ε} or an optimal control u^0. An iteration of the algorithm consists on moving from $\{u, Q_{sup}\}$ to another support control $\{\overline{u}, \overline{Q}_{sup}\}$ such that $J(\overline{u}) \geq J(u)$. The developed algorithm has three procedures:

- the control transformation $u \longrightarrow \overline{u}$;
- the support transformation $Q_{sup} \longrightarrow \overline{Q}_{sup}$;
- the finishing procedure.

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Table of contents

Control transformation

Let $\epsilon \geq 0$ be a given number and $\{u, Q_{sup}\}$ a support control verifying $\beta(u, Q_{sup}) > \epsilon$. We construct another admissible control $\overline{u}(t) = u(t) + \theta \Delta u(t),\ t \in T$, such that $J(\overline{u}) \geq J(u)$, where $\Delta u(t)$ is an ascent direction and $\theta \geq 0$ is the step along this direction. For this, let $\alpha > 0$ and $h > 0$, be the parameters of the algorithm and:

- Construct the sets:

$$T_\alpha = \{t \in T : \ \eta(t) \leq \alpha\}, \quad T_n = T \setminus T_\alpha,$$

$$\text{with } \eta(t) = \min_{j \in J} |E_j(t)|, \ t \in T.$$

- We subdivide T_α into intervals $[\tau_k, \tau^k]$, $k = \overline{1, N}$, $\tau_k < \tau^k \leq \tau_{k+1}$, $T_\alpha = \bigcup_{k=1}^{N} [\tau_k, \tau^k]$, so that $\tau^k - \tau_k \leq h$; $T_{sup} \subset \{\tau_k, k = \overline{1, N}\}$; $u_j(t) = u_{jk} = const,\ t \in [\tau_k, \tau^k],\ k = \overline{1, N}, j \in J$.

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- Then we compute the following quantities:

$$\beta_{jk} = -\int_{\tau_k}^{\tau^k} E_j(t)dt, \qquad q_{jk}(s) = \int_{\tau_k}^{\tau^k} \varphi_j(s,t)dt, \ \ k = \overline{1,N}, \ \ j \in J, \ \ s \in S, \tag{14}$$

$$\beta_{N+1} = -\sum_{j=1}^{r} \int_{T_*} E_j(t)\Delta u_j(t)dt + \sum_{i \in I_{sup}(s), s \in S_{sup}} y_i(s)\overline{\upsilon}_i(s), \tag{15}$$

$$q_{i(N+1)}(s) = \sum_{j=1}^{r} \int_{T_*} \varphi_{ij}(s,t)\Delta u_j(t)dt - \overline{\upsilon}_i(s), i \in I_{sup}(s), \ \ s \in S_{sup}, \tag{16}$$

$$q_{i(N+1)}(s) = \sum_{j=1}^{r} \int_{T_*} \varphi_{ij}(s,t)\Delta u_j(t)dt, \ i \in I_c(s), \ \ s \in S, \tag{17}$$

where

$$\overline{\upsilon}_i(s) = \begin{cases} g_i^*(s) - H_s(i,K)x(t_s), & \text{if } \ y_i(s) < 0, \ i \in I(s), \ \ s \in S, \\ g_{*i}(s) - H_s(i,K)x(t_s), & \text{if } \ y_i(s) > 0, \ i \in I(s), \ \ s \in S, \end{cases} \tag{18}$$

and

$$\Delta u_j(t) = \begin{cases} d_j^+ - u_j(t), & \text{if } \ E_j(t) < -\alpha, \\ d_j^- - u_j(t), & \text{if } \ E_j(t) > \alpha, \qquad t \in T_*, \ \ j = \overline{1,r}. \end{cases} \tag{19}$$

Let us set

$$f_*(I_c(s)) = g_*(I_c(s)) - H(I_c(s),K)x(t_s),$$

$$f^*(I_c(s)) = g^*(I_c(s)) - H(I_c(s),K)x(t_s), \ \ s \in S$$

$$f_*(I_{sup}) = f^*(I_{sup}) = 0;$$

Control transformation

- Using these quantities, we formulate the following support linear problem:

$$\beta' l \longrightarrow \max, \tag{20a}$$

$$f_*(s) \leq \sum_{j=1}^{r} \sum_{k=1}^{N} q_{jk}(s) l_{jk} + q_{N+1}(s) l_{N+1} \leq f^*(s), s \in S, \tag{20b}$$

$$d_j^- - u_{jk} \leq l_{jk} \leq d_j^+ - u_{jk}, \ \ j = \overline{1, r}, \ \ k = \overline{1, N}, \quad 0 \leq l_{N+1} \leq 1. \tag{20c}$$

- Let $\{l^{\varepsilon}, \overline{Q}_B\}$ be an ε-optimal of (20), with $\overline{Q}_B = \{\overline{I}_{sup}, \overline{J}_B, \overline{T}_B\}$.
- the new support control $\{\overline{u}, \widetilde{Q}_s\}$ of the problem (1) is expressed as follows:

$$\overline{u}_j(t) = \begin{cases} u_j(t) + l^{\epsilon}_{jk}, & t \in [\tau_k, \tau^k], \quad k = \overline{1, N}, \\ u_j(t) + l^{\epsilon}_{N+1}\Delta u_j(t), & j = \overline{1, r}, \quad t \in T_*, \end{cases} \tag{21}$$

and $\widetilde{Q}_{sup} = \{\widetilde{I}_{sup}, \widetilde{J}_{sup}, \widetilde{T}_{sup}\}$, $\widetilde{I}_{sup}(s) = \overline{I}_{sup}(s), s \in \widetilde{S}_{sup}$, $\widetilde{J}_{sup} = \{\overline{J}_k, \ k \in \overline{S}_B\}$, $\widetilde{T}_{sup} = \{\tau_k, \ k \in \overline{S}_B\}$.

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Optimality of a new support control

- Let us calculate the new suboptimality estimate $\beta(\overline{u}, \widetilde{Q}_{sup})$.

1. **If** $\beta(\overline{u}, \widetilde{Q}_{sup}) \leq \epsilon$, then $\overline{u}$ is an ϵ-optimal control for the problem (1);
2. **Otherwise**, we perform either a new iteration with a support control $\{\overline{u}, \widetilde{Q}_{sup}\}$ and parameters $\overline{\alpha} < \alpha$, $\overline{h} < h$, or we do the change of the support.

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Table of contents

Change of support

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Let $\{\overline{u}, \widetilde{Q}_{sup}\}$ be the support control found after the resolution of the problem (20). Calculate with the formula (9) the cocontrol $\widetilde{E}'(t) = -\widetilde{\psi}'(t)B - c_2'(t),\ t \in T$, corresponding to $\{\overline{u}, \widetilde{Q}_{sup}\}$. After that, we construct the quasicontrol $w(t),\ t \in T$:

$$w_j(t) = \begin{cases} d_j^-, & \text{if } \widetilde{E}_j(t) > 0, \\ d_j^+, & \text{if } \widetilde{E}_j(t) < 0, \\ \in [d_j^-, d_j^+], & \text{if } \widetilde{E}_j(t) = 0, \quad j = \overline{1, r}, \quad t \in T, \end{cases} \quad (22)$$

and the corresponding quasitrajectory $\chi(t),\ t \in T$:

$$\dot{\chi} = A\chi + Bw + r, \quad \chi(0) = x_0. \quad (23)$$

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- Then compute the vectors:

$$\gamma(\widetilde{J}_{sup}, \widetilde{T}_{sup}) = \widetilde{\varphi}_{sup}^{-1}\left(g_*^*(\widetilde{I}_{sup}(s)) - H(\widetilde{I}_{sup}(s), K)\chi(t_s), s \in S_{sup}\right),$$

$$\gamma_i^*(s) = \sum_{j \in \widetilde{J}_k, k \in \widetilde{K}_{sup}} \varphi_{ij}(s, t_k)\gamma(j, t_k) + H_s(i, K)\chi(t_s) - g_i^*(s), \quad i \in \widetilde{I}_c(s)$$

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where

$$g_{*i}^*(s) = \begin{cases} g_{*i}(s), & \text{if } \widetilde{y}_i(s) < 0, \\ g_i^*(s), & \text{if } \widetilde{y}_i(s) > 0, \end{cases} \tag{24}$$

Change of support

- **If** :

$$||\gamma(\widetilde{J}_{sup}, \widetilde{T}_{sup})|| \leq \mu, \quad (25)$$

$$\gamma^*(\widetilde{I}_c(s)) \geq 0, \;\; \gamma_*(\widetilde{I}_c(s)) \leq 0, \;\; s \in S, \quad (26)$$

are verified, then we perform the finishing procedure with a support $\overline{Q}_{sup} = \widetilde{Q}_{sup}$.

- **Otherwise,** we will change the support ($\widetilde{Q}_{sup} \longrightarrow \overline{Q}_{sup}$) with an iteration of the dual method.

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Table of contents

Finishing procedure

- The finishing procedure consists to construct an optimal support $Q^*_{sup} = \{I^*_{sup}, J^*_{sup}, T^*_{sup}\}$ in order to have

$$g_*(s) \le H(s)\chi(t_s) \le g^*(s), \quad s \in S.$$

- The support Q^*_{sup} is determined by solving the following equations:

$$\sum_{j\in\bar{J}_k}\sum_{k\in\bar{K}_{sup}} (d_j^+ - d_j^-) sign\dot{E}_j(t_k) \int_{t_k}^{V_k(T^*_{sup})} \varphi_{ij}(s,t)dt - \quad (27)$$

$$g^*_{*i}(s) + H_s(i,K)\chi(t_s) = 0, \quad i \in \bar{I}_{sup}(s), s \in \bar{S}_{sup},$$

where $V_k(T^*_{sup}), k \in K^*_{sup}$, verifies the relations :

$E_j(V_k(T^*_{sup}), T^*_{sup}) = 0, \quad V_k(\bar{T}_{sup}) = t_k, \quad j \in \bar{J}_k, \quad k \in \bar{K}_{sup};$

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Finishing procedure

- We solve the equation (27) with Newton method. For each approximation, if the conditions (26) are not verified, then we change the support via the dual method in order to get the relation (26).
- Let $Q^*_{sup} = \{I^*_{sup}, J^*_{sup}, T^*_{sup}\}$ be a solution of the system (27). Then the quasicontrol $w^*(t), \quad t \in T$, calculated with the support Q^*_s, (22) and (23) is an admissible and optimal control of the problem (1).

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Merci pour votre attention!